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Mixed regularization methods for the Cauchy problems of the Helmholtz equation

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چکیده

In this paper, two mixed filter functions are applied for the regularization of the Helmholtz equation. Bound of error is presented under a priori bound respect to the exact solution. At last, the numerical experiment shows the efficiency and effectiveness of these methods.

Keywords: Ill-posed, Helmholtz equation, Regularization method, Cauchy problem, mixed filter function

Introduction

It is well known that inverse problems of elliptic equations are ill-posed in the sense that small noise in input data can introduce arbitrarily large error in the solution [۲۲]. For continuous dependence of the solution on the Cauchy data for satisfying Third Hadamard well-posedness condition, additional conditions are needed. These conditions are bounds for the norm of the solution and noise.

Class of Regularization methods includes a lot of various methods for minimization of noise effects. The classical methods such as, Tikhonov method, truncate singular value decomposition (TSVD), L-curve method and iterative methods can be found in [۷-۱۱].

In the literature inverse problem of the Cauchy problem of Helmholtz equation solved by some numerical methods such as separation of variable, spectral methods, boundary element method (BEM) and solution is severely ill-posed in large scale problems. Almost, the solution is sensitive respect to noise norm. Regularization methods are approached to decrease this affectability.

To vanish ill-posedness of this problem different regularization methods applied by Xiao-Li Feng et al [۳], Hao Cheng et al [٤], H.H. Qin and T. Wei [٥], Xiangtuan Xionget al [٦], Haihua Qin and Jingmei Lu [۲٥]. In classical methods, accuracy is low and for new iterative methods the computational cost is high.

In this paper, two filter functions are utilized from mixing (TSVD), exponential functions and Tikhonov filter. To decrease the truncation error in numerical implementation of Tikhonov method in unbounded problems, problems with a large variance in a spectrum similar to Cauchy problems of Helmholtz and Laplace equations, we can apply TSVD-exponential filtering by using of TSVD method in one part of the spectrum and exponential filtering in another part.

۱. Mathematical problem

We consider Cauchy problem for the Helmholtz equation with Cauchy conditions:

$$\Delta w(x,y) + k^{۲} w(x,y) = ۰, \quad ۰ < x < \pi\,; ۰ < y < ۱ \qquad (۱.۱)$$

$$w(x,۰) = \varphi(x), \qquad ۰ \le x \le \pi$$

$$w_y(x,۰) = \psi(x), \qquad ۰ \le x \le \pi$$

$$w(۰,y) = w(\pi,y) = ۰, \qquad ۰ \le y \le ۱$$

According to the linearity of the problem (۱,۱), we divide it into the following two ill-posed problems:

$$\Delta u(x,y)+k^{۲}u(x,y)=۰,\qquad ۰<x<\pi\,;\,۰<y<۱ \tag{۱.۲}$$
$$u(x,۰)=\varphi(x),\qquad ۰\le x\le\pi$$
$$u_y(x,۰)=۰,\qquad ۰\le x\le\pi$$
$$u(۰,y)=u(\pi,y)=۰,\qquad ۰\le y\le ۱$$

and

$$\Delta v(x,y)+k^{۲}v(x,y)=۰,\qquad ۰<x<\pi\,;\,۰<y<۱ \tag{۱.۳}$$
$$v(x,۰)=۰,\qquad ۰\le x\le\pi$$
$$v_y(x,۰)=\psi(x),\qquad ۰\le x\le\pi$$
$$v(۰,y)=v(\pi,y)=۰,\qquad ۰\le y\le ۱$$

Where $w=u+v$ is solution of the problem (۱،۱). In problem (۱،۲) the positive real number k is the wave number, the noisy data $\varphi^{\delta}(x)$ instead of Cauchy data is available which satisfies:

$$\|\varphi_{\delta}-\varphi\|\le\delta \tag{۱.۴}$$

Suppose that the a priori bound is:

$$\max\|u(:,۱)\|\le E \tag{۱.۵}$$

Spectral Galerkin Method (SGM) leads to a solution of problem (۱،۲) as follows [٦]:

$$u(x,y)=\sum_{n=۱}^{\infty}\langle\sin(nx),\varphi(x)\rangle\sin(nx)\cosh\left(\sqrt{n^{۲}-k^{۲}}y\right) \tag{۱.٦}$$

where $\langle .,.\rangle$ denotes the inner product in $L^{۲}(۰,\pi)$ which is defined by:

$$\langle\sin(nx),\varphi(x)\rangle=\frac{۲}{\pi}\int_{۰}^{\pi}\varphi(t)\sin(nt)\,dt \tag{۱.٧}$$

If $n\le k$, then $\cosh\left(\sqrt{n^{۲}-k^{۲}}y\right)=\cos\left(\sqrt{k^{۲}-n^{۲}}y\right)$ and singular values of the problem is small and if $n\gg k$ the singular values are exponentially and multiple with noisy data. For overcoming to noise propagation the introduce the regularization methods.

۲. Regularization methods

Optimality of the regularization methods theoretically is presented in [۱] and recently application of regularizations methods are: iterative methods [۲۳],L-curve method [۱٦], landweber method [۱٤], TSVD method [۱۸], optimizational method [۲۰], variational method [۲۱].

Several methods for elliptic equations are presented in [۱۷, ۱۹, ۲٤] and for this problem some methods are applied such as regularization with posterior parameter [۲], Tikhonov method [۱٥]. Some advantages of these methods are shown in [۱۲-۱٤].

For ill-posed inverse problem $Ax = b$ regularization method is defined as:

$$R_\alpha \stackrel{\text{def}}{=} g_\alpha(A^*A)A^* \qquad (۲.۱)$$

Where A^* is the adjoint operator of A .
Filter functions are class of functions which satisfy:

$$\lim_{\alpha\to ۰} \phi_\alpha(\lambda) = ۱ \qquad (۲.۲)$$

The most popular regularization methods by filtering are Tikhonov and TSVD methods. In these methods inverse operators are defined by filter functions:

$$g_\alpha(\lambda) = \frac{۱}{\lambda}\phi_\alpha(\lambda) \qquad (۲.۳)$$

For $\phi_\alpha(\lambda) = ۱$ the inverse operator is ill-condition and solution is naive. Selection of α can be helpful to avoid noise propagation.

At first, we will consider TSVD filter function for this problem:

$$\phi_\alpha(\lambda) = \begin{cases} ۱ & \lambda \geq \alpha \\ ۰ & \lambda < \alpha \end{cases} \qquad (۲.۴)$$

and

$$g_\alpha(\lambda) = \begin{cases} \frac{۱}{\lambda} & \lambda \geq \alpha \\ ۰ & \lambda < \alpha \end{cases} \qquad (۲.۵)$$

where $\lambda = ۱/\cosh(\sqrt{n^۲ - k^۲}y)$ is the singular value of the inverse problem.

Supposex^*, x_α, x^δdenote exact solution, regularization solution and noisy solution of the problem, respectively. The truncation error function (TEF) is defined as:

$$r_\alpha(\lambda) = ۱ - \phi_\alpha(\lambda) \qquad (۲.۶)$$

The truncation error depends on (TEF):

$$x^* - x_\alpha = r_\alpha(A^*\mathrm{A})x^* \qquad (۲.۷)$$

In Tikhonov method we choose the filter function as:

$$\phi_\alpha(\lambda) = \frac{\lambda^۲}{\lambda^۲ + \alpha^۲} \tag{۲.۸}$$

This filter is the cumulative distribution function (CDF) of the gaussian (normal) random variable
and

$$g_\alpha(\lambda) = \frac{۱}{\lambda}\phi_\alpha(\lambda) = \frac{\lambda}{\lambda^۲ + \alpha^۲} \tag{۲.۹}$$

Is the probability distribution function (PDF) of the Gaussian (normal) random variable. Tikhonov regularization is equal to normalization of the singular value. Equivalently, we can use the exponential form of these filters:

$$\phi_\alpha(\lambda) = \left(۱ - e^{-\frac{\lambda}{\alpha}}\right) \tag{۲.۱۰}$$

$$g_\alpha(\lambda) = \frac{۱}{\lambda}\phi_\alpha(\lambda) = \frac{۱}{\lambda}\left(۱ - e^{-\frac{\lambda}{\alpha}}\right) \tag{۲.۱۱}$$

For exponential random variable (CDF) and (PDF):

$$\phi_\alpha(\lambda) = e^{-\frac{\alpha}{\lambda}} \tag{۲.۱۲}$$

$$g_\alpha(\lambda) = \frac{۱}{\lambda}\phi_\alpha(\lambda) = \frac{۱}{\lambda}e^{-\frac{\alpha}{\lambda}} \tag{۲.۱۳}$$

By this approach we can build different filter functions with considering (CDF) and (PDF) of random variables. But, filter structure should be similar to distribution of the singular values. In the Cauchy problem of Helmholtz equation, the singularity is exponential and exponential filters can be helpful.

Method (I): By use of TSVD method for the first part of the spectrum, we can select the truncation parameter N from (۲,۵) as follows:

$$\lambda = ۱/cosh\left(\sqrt{n^۲ - k^۲}y\right) \geq \alpha \Rightarrow N = \left[\sqrt{(arccosh(\alpha))^۲ + k^۲}\right] \tag{۲.۱۴}$$

And in the another part the filter function is equal to the exponential function (۲،۱۱).The solution of problem (۱،٦) has the form as:

$$u(x,y) = \sum_{n=۱}^{N} \langle sin(nx), \varphi(x)\rangle \, sin(nx)\, cosh\left(\sqrt{n^{۲} - k^{۲}}y\right) + \sum_{n=N+۱}^{\infty} \langle sin(nx), \varphi(x)\rangle \, sin(nx)\, cosh\left(\sqrt{n^{۲} - k^{۲}}y\right) \left(۱ - e^{-۱/\alpha\, cosh\left(\sqrt{n^{۲}-k^{۲}}y\right)}\right) \tag{۲.۱۵}$$

Method (II): From (۲،۱۳) and similar approach, we get:

$$u(x,y) = \sum_{n=۱}^{N} \langle \sin(nx), \varphi(x)\rangle \sin(nx) \cosh\left(\sqrt{n^{۲} - k^{۲}}y\right) + \sum_{n=N+۱}^{\infty} \langle \sin(nx), \varphi(x)\rangle \sin(nx) \cosh\left(\sqrt{n^{۲} - k^{۲}}y\right)\ e^{-\alpha \cosh\left(\sqrt{n^{۲}-k^{۲}}y\right)} \tag{۲.۱۶}$$

۳. Error bounds

The error bound for regularization method (I) is obtained as follow :

Lemma (۳،۱): Suppose that u is the exact solution and u^{δ} is the regularization solution by method (I) for problem (۱،۲) also will two conditions (۱،٤) and (۱،٥) satisfying. If we select $\alpha = (\frac{\delta}{E})^{y}$, then for fixed $۰ < y < ۱$ we get the error bound:

$$\left\|u(x,y) - u^{\delta}(x,y)\right\| = \delta^{۱-y} E^{y} (۱ + \frac{۲\delta}{\|\varphi\|}) \tag{۳.۱}$$

Proof: We compute bound of error in two parts of spectrum:

If $n \le N$, then

$$\left\|u_N(x,y) - u_N^{\delta}(x,y)\right\|^{۲} = \frac{\pi}{۲} \sum_{n=۱}^{\infty} \left|\langle \varphi^{\delta} - \varphi, \sin(nx)\rangle\right|^{۲} \cosh^{۲}\left(\sqrt{n^{۲} - k^{۲}}y\right) \le \frac{\delta^{۲}}{\alpha^{۲}}$$

$$\left\|u_N(x,y)-u_N^{\delta}(x,y)\right\| \leq \frac{\delta}{\alpha}= \delta^{۱-y} E^{y} \qquad (۳.۲)$$

If $n>N$, then

$$\left\|u\,(x,y)-u_N^{\delta}(x,y)\right\|^{۲} =$$

$$\left\|u\,(x,y)-u_N^{\delta}(x,y)\right\|^{۲} = \frac{\pi}{۲}\sum_{n=۱}^{\infty}\left|\langle \varphi^{\delta}-\varphi, \sin(nx)\rangle\right|^{۲} \cosh^{۲}\left(\sqrt{n^{۲}-k^{۲}}y\right)\left(e^{-\frac{۲}{\alpha\cosh\left(\sqrt{n^{۲}-k^{۲}}y\right)}}\right) \leq$$

$$\sup_{n>N}\frac{\cosh^{۲}\left(\sqrt{n^{۲}-k^{۲}}y\right)}{\cosh^{۲}\left(\sqrt{n^{۲}-k^{۲}}\right)}\left(e^{-\frac{۲}{\alpha\cosh\left(\sqrt{n^{۲}-k^{۲}}y\right)}}\right)E^{۲} \times \frac{\left|\langle \varphi-\varphi^{\delta}, \sin(nx)\rangle\right|^{۲}}{\left|\langle \varphi, \sin(nx)\rangle\right|^{۲}}$$

Since in this part of the spectrum $\frac{\cosh\left(\sqrt{n^{۲}-k^{۲}}y\right)}{\cosh\left(\sqrt{n^{۲}-k^{۲}}\right)} \leq ۲e^{-\sqrt{n^{۲}-k^{۲}}(۱-y)}$ and $\alpha = (\frac{\delta}{E})^{y}$, we get:

$$\left\|u\,(x,y)-u_N^{\delta}(x,y)\right\| =$$

$$\sup_{n>N} ۲e^{-\sqrt{n^{۲}-k^{۲}}(۱-y)}\left(e^{-\frac{۱}{\alpha\cosh\left(\sqrt{n^{۲}-k^{۲}}y\right)}}\right)E\,\frac{\delta}{\|\varphi\|} \leq \frac{۲\delta^{۲-y}E^{y}}{\|\varphi\|} \qquad (۳.۳)$$

Adding (۳,۲) and (۳,۳) yields (۳,۱).

For regularization method (II) the error bound satisfies in (۳,۱) and for problem (۱,۳) with Cauchy data ψ bound of error is similar. In more general problem (۱,۱) bound of error for both methods is utilize from summation of these bounds.

Lemma (۳,۲): Suppose that w and w^{δ} is the exact and regularization solution by each of methods (I) and (II) for problem (۱,۱) respectively, and two conditions (۱,٤) and (۱,٥) are satisfied. If we select $\alpha = (\frac{\delta}{E})^{y}$, then for fixed $۰ < y < ۱$ we get the error bound:

$$\left\|w(x,y)-w^{\delta}(x,y)\right\| = ۲\delta^{۱-y}E^{y}(۱+\frac{۲\delta}{\|\varphi\|}) \qquad (۳.۴)$$

٤. Numerical tests

In this section, some numerical examples are presented for verifying the validity of the proposed methods. To compare the exact and approximate solutions root mean square relative error is used:

$$RMSRE = \frac{\sqrt{\frac{۱}{M^{۲}}\sum_{i=۱}^{M}\sum_{j=۱}^{M}(u^{\delta}(x_i,y_j)-u(x_i,y_j))^{۲}}}{\sqrt{\frac{۱}{M^{۲}}\sum_{i=۱}^{M}\sum_{j=۱}^{M}(u(x_i,y_j))^{۲}}}$$

where $x_i = \frac{\pi}{M}(i-۱),\ y_j = \frac{۱}{M}(j-۱),\ i = ۱,\dots,M,\ j = ۱,\dots,M\ ,M = ۳۰$

Test. ۱) We consider the following Cauchy problem:

$$\Delta u(x,y) + ku(x,y) = ۰, \qquad ۰ < x < \pi\,;\, ۰ < y < ۱ \qquad (۴.۱)$$

$$u(x,۱) = x^{۲}(\pi - x) + \sin(۲kx)\cosh\left(\sqrt{۳}k\right), \qquad ۰ \le x \le \pi$$

$$u_y(x,۰) = ۰, \qquad ۰ \le x \le \pi$$

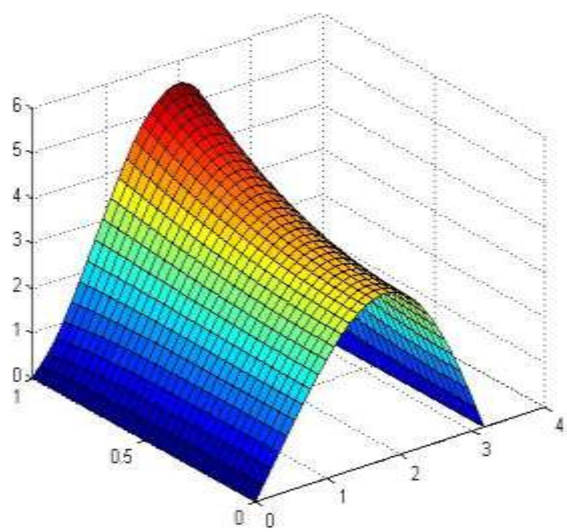
$$u(۰,y) = u(\pi,y) = ۰, \qquad ۰ \le y \le ۱$$

We select

$$\varphi(x) = u(x,۰) = \sum_{n=۱}^{m}\langle u(x,۱),\sin(nx)\rangle \sin(nx)\frac{۱}{\cosh(\sqrt{n^{۲}-k^{۲}})}$$

as Cauchy data for problem (٤،۱), with m=٣٦, ε denotes the error level and noisy Cauchy data is $\varphi^{\delta} = \varphi + \varepsilon(x-۲)(۴-x)\sin(x)$.

In Fig۱ we show exact solution and the error between the regularized solution (I) and the exact solution. In Fig۲ we show the numerical results at $y = ۱\ , k = ۰.۵$ for $\varepsilon = ۱۰^{-۳}, ۱۰^{-۲}, ۵\times۱۰^{-۲}$



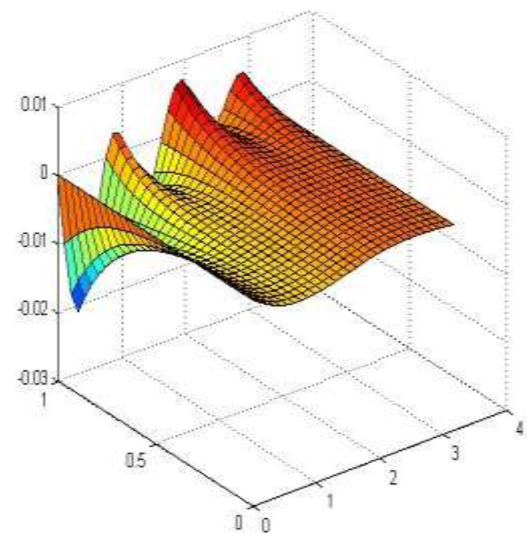


Fig ١. Exact solution u for k= ٠,٥

$u^{\delta} - u$ with $\varepsilon =$ ١٠ $^{-٣}$

Table ١. RMSRE with k= ٠,٥ for method (I)

	10^{-4}	10^{-3}	10^{-2}	5×10^{-2}
$RMSRE$	٠.٠٠٠٢٠٣۴٩	٠.٠٠٠۵٩١٢٩	٠.٠٠۵٠	**0.0281**

Table ٢. RMSRE with k= ٠,٥ for method (II)

	10^{-4}	10^{-3}	10^{-2}	5×10^{-2}
$RMSRE$	٠.٠٠٠٢٩۴۵٢	٠.٠٠٠۶٩٨٣٨	٠.٠٠۴۶	**0.0287**

The relative root mean square error between the regularized solution methods (I), (II) and the exact solution is shown in Tables ۱,۲.

Test. ۲) For Cauchy problem:

$$\Delta u(x,y) + ku(x,y) = \cdot, \qquad \cdot < x < \pi \,; \cdot < y < ۱ \tag{۴.۲}$$

$$u(x,\cdot) = \varphi(x) = \sin(x), \qquad \cdot \le x \le \pi$$

$$u_y(x,\cdot) = \cdot, \qquad \cdot \le x \le \pi$$

$$u(\cdot,y) = u(\pi,y) = \cdot, \qquad \cdot \le y \le ۱$$

the exact solution is:

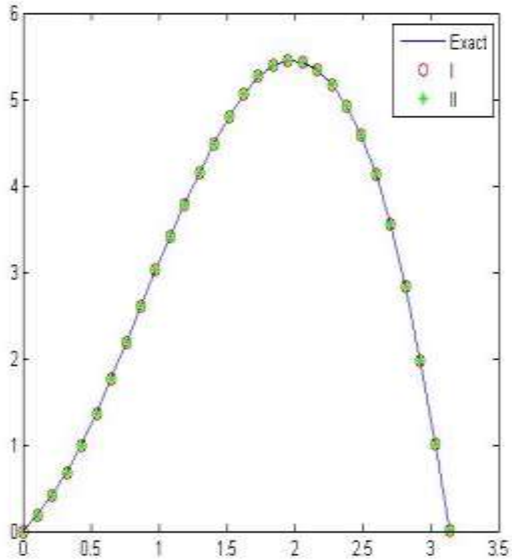
$$u(x,y) = \cosh\left(\sqrt{۱ - k^{۲}}y\right)\sin(x) \tag{۴.۳}$$

Suppose that the noisy version Cauchy data is $\varphi^{\delta} = \varphi + \varepsilon\, randn(size(\varphi))$ and M=۵۰.

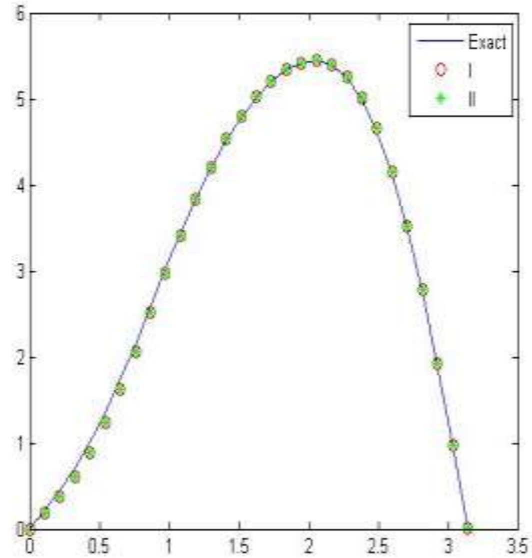
In Fig۳ we show exact and regularization solution for methods (I) and (II) for k=۰,۳, ۲,۷ and $\varepsilon = ۱۰^{-۲}$ at y=۰,۹. the errors for k=۲,۷ and y=۰,۵, ۰,۹ are shown in Table ۳.

Table ۳. k=۲,۷, $\varepsilon = ۰.۱$

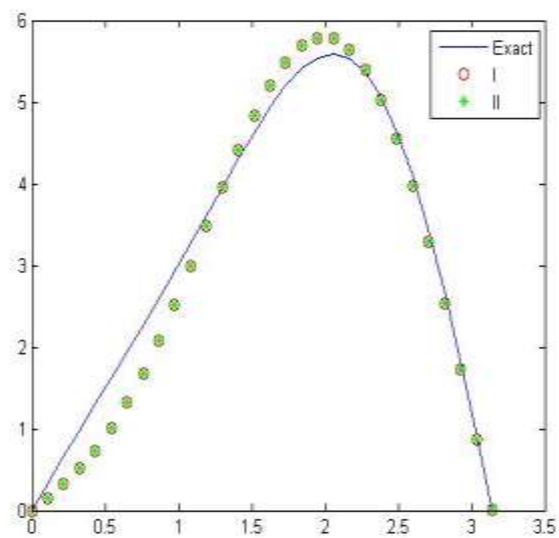
y	0.5	0.9
$RMSRE\ (I)$	۰.۰۱۲۰	۰.۰۰٦٤
$RMSRE\ (II)$	۰.۰۱۲۲	۰.۰۰٦٦



(a) $\varepsilon = ۱۰^{-۳}$

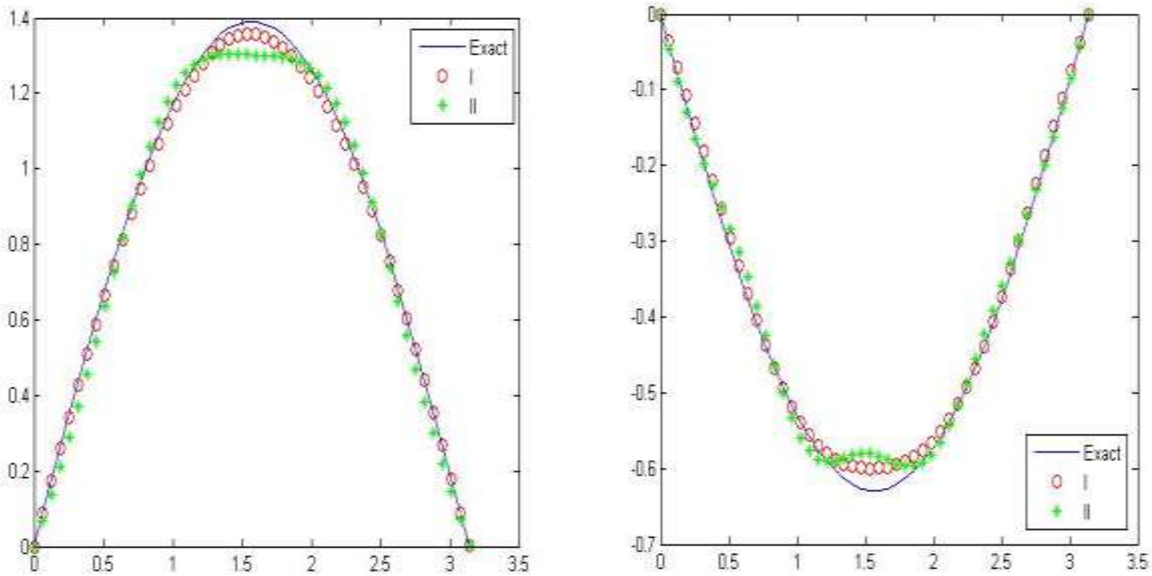


(b) $\varepsilon = ۱۰^{-۲}$



(c) $\varepsilon = ۵\times ۱۰^{-۲}$

Fig ۲. K=۰.۵, y=۱



k= ۰,۳, y= ۰,۹ 			k= ۲,۷, y= ۰,۹

Fig ۳. $\varepsilon = ۱۰^{-۲}$

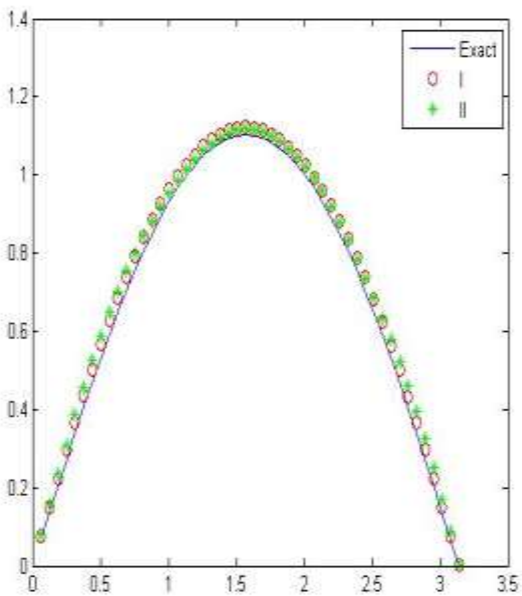
Test. ۳) The exact solution of the problem (۱,۱) with $\varphi = \psi = \sin(\sqrt{۲}kx)$ and $k = \sqrt{۲}, \frac{۱}{\sqrt{۲}}$ is:

$$u(x,y) = \sin(\sqrt{۲}kx)\left(\cosh(ky) + \frac{\sinh(ky)}{k}\right)$$

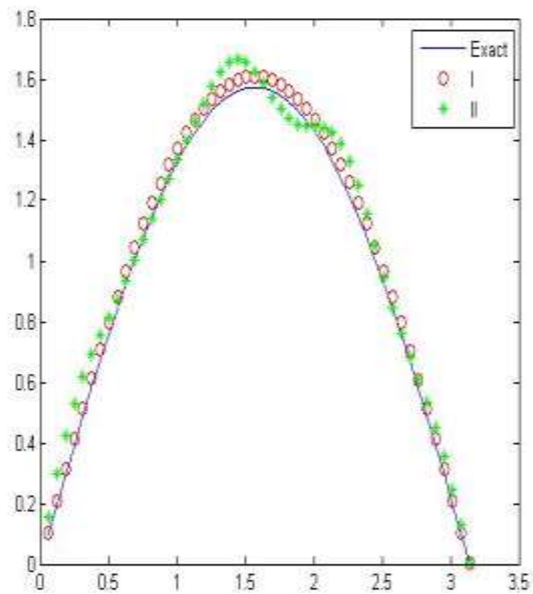
The numerical results for $k = \frac{۱}{\sqrt{۲}}, \varepsilon = ۵ \times ۱۰^{-۲}, y = ۰.۱, ۰.۵, ۰.۸, ۱$ are shown in Fig٤. and the errors between the exact and regularized solutions for $k = \frac{۱}{\sqrt{۲}}, \varepsilon = ۵ \times ۱۰^{-۲}$ are shown in Table ٤

Table ٤. $k = \frac{١}{\sqrt{٢}}, \varepsilon = ٥\times ١٠^{-٢}$

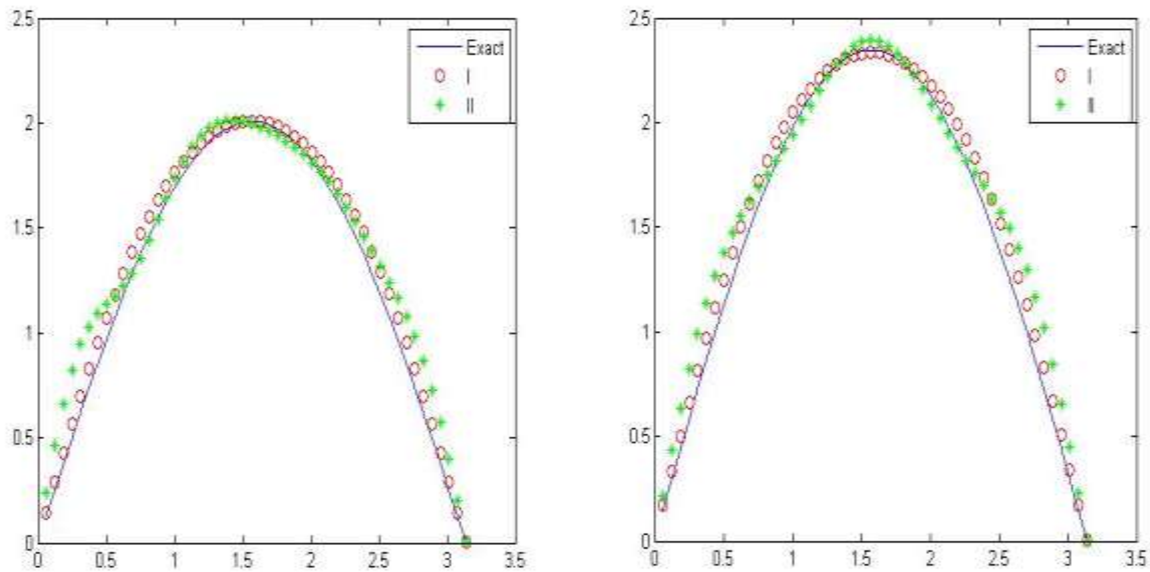
y	0.1	0.3	0.5	0.7	0.9
$RMSRE(I)$	۰.۰۴۰۷	۰.۰۳۹۱	۰.۰۳۳۶	۰.۰۲۸۱	**0.0173**
$RMSRE(II)$	۰.۰۴۲۹	۰.۰۴۰۰	۰.۰۳۴۸	۰.۰۳۱۱	**0.0179**



(a)



(b)



(c) (d)

Fig ٤. $k = \frac{١}{\sqrt{٢}}, \varepsilon = ٥ \times ١٠^{-٢}$ in the (a)$y = ٠.١$ (b))$y = ٠.٥$ (c))$y = ٠.٨$ (d))$y = ١$

Conclusion

For both methods the errors are the same. For problems with exponential singularity use of TSVD-exponential filters are efficient. For decreasing truncation error use of exponential filters in truncated domain are effectiveness.

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